HW5

1.a

$x(t) = 30\cos(\theta(t))$, $\theta(t) = 10^6 t + 4\sin(15t)$

$f_i(t) = \frac{1}{2\pi}\frac{d}{dt}\theta(t) = \boxed{\frac{1}{2\pi}\left(10^6 + 60\cos(15t)\right)}$

using Carson's rule (other estimates are OK too):

$BW \approx 2\left(\frac{f_\Delta}{W} + 1\right)W = 2(f_\Delta + W)$ (W is the bandwidth of the message, $\frac{15}{2\pi}$ Hz)

$= 2\left(\frac{60}{2\pi} + \frac{15}{2\pi}\right) = \boxed{23.87 \text{ Hz}}$

1.b

$y(t) = 30\cos(5\theta(t))$

$f_i(t) = \frac{1}{2\pi}\frac{d}{dt}5\theta(t) = \boxed{\frac{5}{2\pi}\left(10^6 + 60\cos(15t)\right)}$

$BW \approx 2(f_\Delta + W) = 2\left(\frac{5\cdot 60}{2\pi} + \frac{15}{2\pi}\right) = \boxed{100 \text{ Hz}}$

1.c

$k_f m(t) = f_i - f_c = \boxed{\frac{60}{2\pi}\cos(15t)}$

I realized later that this is ambiguous, so it's OK if you did it for $y(t)$ instead:

$k_f m(t) = f_i - f_c = \boxed{\frac{300}{2\pi}\cos(15t)}$

1.d

$k_p m(t) = \int (f_i - f_c)\,dt = \boxed{\frac{4}{2\pi}\sin(15t)}$

or

$k_p m(t) = \boxed{\frac{20}{2\pi}\sin(15t)}$

1.e

No units are given, so we just find the $(\text{rms})^2$. This is because $f_c >> f_\Delta$, so the signal(s) is essentially just sinusoidal.

$P_x = P_y = \boxed{\frac{30^2}{2}}$

2.a)

$f_{c1} = \boxed{3\,MHz}$, $f_{c2} = 7\cdot f_{c1} = \boxed{21\,MHz}$, $f_{c3} = 11\cdot f_{c2} = \boxed{231\,MHz}$

2.b)

$f_{\Delta 1} = \boxed{500kHz}$, $f_{\Delta 2} = 7\cdot f_{\Delta 1} = \boxed{3.5\,MHz}$, $f_{\Delta 3} = 11\cdot f_{\Delta 2} = \boxed{38.5\,MHz}$

Note: I accidentally typed "kHz" instead of just "Hz". f_Δ is probably too big compared to f_c to be realistic. We can still go through the calculations though.

2.c)

$BW_{x1} = 2(f_{\Delta 1} + W) = 2(500k + 20k) = \boxed{1040\,kHz}$

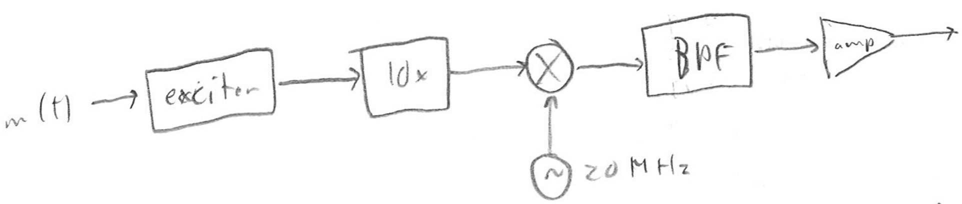
$BW_{x2} = 2(f_{\Delta 2} + W) = 2(3.5M + 20k) = \boxed{7040\,kHz}$

$BW_{x3} = 2(f_{\Delta 3} + W) = 2(38.5M + 20k) = \boxed{77040\,kHz}$

3.a)

$f_{\Delta i} = 10\,kHz$, $f_{\Delta o} = 100kHz$ → we need a 10x frequency multiplier

$f_{ci} = 8MHz$, $f_{co} = 60MHz$ But $8\cdot 10$ is 80, so we need to shift the signal from the multiplier down by 20 MHz.



The filter would have a center frequency of 60MHz and a BW between 2(100k + 0) = 200kHz and 80 + 20 = 100 MHz. It would probably be easiest to use a BW near the high end.

4.)

$W = 15\,kHz$

$BW = 2(f_\Delta + W)$

f_Δ (kHz)	BW (kHz)
0.1	30.2
0.5	31
1	32
5	40
10	50
50	130
100	230

HW 5

5)

$$\frac{f_\Delta}{W} = 5 \rightarrow f_{\Delta\,music} = 5 \cdot W_{music} = 5 \cdot 15\,kHz = 75\,kHz$$

$BW_{music} = 2(75 + 15) = 180\,kHz$

$$\frac{BW_{talk}}{BW_{music}} = \frac{60}{180} = \boxed{33.3\%}$$

$f_{\Delta\,talk} = 5 \cdot W_{talk} = 25\,kHz$

$BW_{talk} = 2(25 + 5) = 60\,kHz$

6.a) $\bar{x} = E\{x\} = \mu_x$

$cov(X,Y) = E\{(X - \bar{X})(Y - \bar{Y})\} = E\{XY - X\bar{Y} - Y\bar{X} + \bar{X}\bar{Y}\}$

$= \overline{XY} - \bar{X}\bar{Y} - \bar{Y}\bar{X} + \bar{X}\bar{Y} = \overline{XY} - \bar{X}\bar{Y}$

If X and Y are independent, then $\overline{XY} = \bar{X}\bar{Y}$, so $\boxed{cov(X,Y) = 0}$

6.b)

$Y = \alpha X + \beta \rightarrow \bar{Y} = E\{\alpha X + \beta\} = \alpha\bar{X} + \beta$

$\overline{XY} = E\{(\alpha X + b)X\} = E\{\alpha X^2 + \beta X\} = \alpha\overline{X^2} + \beta\bar{X}$

$cov(X,Y) = \alpha\overline{X^2} + \beta\bar{X} - \alpha\bar{X}^2 - \beta\bar{X} = \alpha\overline{X^2} - \alpha\bar{X}^2 = \boxed{\alpha(\overline{X^2} - \bar{X}^2) = \alpha\sigma_x^2}$

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$\varepsilon^2 = E\{(Y - \tilde{Y})^2\}$

$(Y - \tilde{Y})^2 = (Y - \alpha X - b)^2 = Y^2 - \alpha XY - \beta Y - \alpha XY + \alpha^2 X^2 + \alpha\beta X - \beta Y + \alpha\beta X + \beta^2$

$= Y^2 - 2\alpha XY - 2\beta Y + \alpha^2 X^2 + 2\alpha\beta X + \beta^2$

$\varepsilon^2 = \overline{Y^2} - 2\alpha\overline{XY} - 2\beta\bar{Y} + \alpha^2\overline{X^2} + 2\alpha\beta\bar{X} + \beta^2$

Find the critical points by finding where the first partial derivatives are 0.

$\frac{\partial}{\partial\alpha}\varepsilon^2 = -2\overline{XY} + 2\alpha\overline{X^2} + 2\beta\bar{X} = 0$

$\frac{\partial}{\partial\beta}\varepsilon^2 = -2\bar{Y} + 2\alpha\bar{X} + 2\beta = 0$

$$\rightarrow \begin{bmatrix} 2\overline{X^2} & 2\bar{X} \\ 2\bar{X} & 2 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 2\overline{XY} \\ 2\bar{Y} \end{bmatrix}$$

7 (cont.)

We can use row reduction (Gaussian elimination), matrix inverse, etc. to find α and β.

I used Mathematica to do it for me.

$$\boxed{\alpha = \frac{\overline{XY} - \bar{X}\bar{Y}}{\overline{X^2} - \bar{X}^2} = \frac{\mathrm{Cov}(X,Y)}{\sigma_x^2}}$$

$$\boxed{\beta = -\frac{\bar{X}\,\overline{XY} - \overline{X^2}\,\bar{Y}}{\overline{X^2} - \bar{X}^2}}$$

with a bit of effort, β can be simplified

$$\bar{X}\,\overline{XY} - \overline{X^2}\,\bar{Y} = \bar{X}(\overline{XY} - \bar{X}\bar{Y} + \bar{X}\bar{Y}) - \overline{X^2}\,\bar{Y} = \bar{X}(\overline{XY} - \bar{X}\bar{Y}) + \bar{X}^2\bar{Y} - \overline{X^2}\,\bar{Y}$$

$$= \bar{X}\,\mathrm{cov}(X,Y) - \bar{Y}(\overline{X^2} - \bar{X}^2) = \bar{X}\,\mathrm{cov}(X,Y) - \bar{Y}\sigma_x^2$$

$$\beta = -\frac{\bar{X}\,\mathrm{cov}(X,Y) - \bar{Y}\sigma_x^2}{\sigma_x^2} = -(\bar{X}\alpha - \bar{Y}) = \boxed{\bar{Y} - \alpha\bar{X}}$$

Finally, we need to check that the critical point is actually a minimum with the 2nd partial derivative test.

$$H = \left(\frac{\partial^2}{\partial\alpha^2}\varepsilon^2\right)\left(\frac{\partial^2}{\partial\beta^2}\varepsilon^2\right) - \left(\frac{\partial^2}{\partial\alpha\partial\beta}\varepsilon^2\right)^2$$

$$= (2\overline{X^2})(2) - (2\bar{X})^2 = 4(\overline{X^2} - \bar{X}^2) = 4\sigma_x^2 > 0$$

$$\frac{\partial^2}{\partial\alpha^2}\varepsilon^2 = 2\overline{X^2} > 0$$

So the point we found must be a minimum.